

ON THE NATURE OF THE ABELIAN HIGGS MODEL PHASE TRANSITION

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ABSTRACT

The nature of the Abelian Higgs Model phase transition is investigated. A variational approximation is used in the evaluation of the relevant finite temperature effective potential. Some of the results presented are valid not only in the Abelian Higgs Model, but also in more complex theories.

Several recent studies¹⁻⁶ have been devoted to the phase transition of the Abelian Higgs Model. Besides being interesting in its own right, the Abelian Higgs Model gives a simple setting in which one can develop techniques that might be useful in the investigation of other gauge theories with spontaneous symmetry breaking. In this lecture I discuss a variational technique that can be used in the evaluation of the finite temperature effective potential, which is an important tool in the investigation of phase transitions. I illustrate this technique by studying the daisy and superdaisy resummed^{7,8,9} finite temperature effective potential of the Abelian Higgs Model.

As discussed in Ref.8, using the composite operator method^{7,8,10} one can show that, if $e^3 < \lambda < e^2$, the daisy and superdaisy resummed effective potential of the Abelian Higgs Model can be written as

$$V_T^{res} = V_T^{res}(\phi, G_0) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{2}\oint_k \ln G_0^{-1}(k) + \frac{1}{2}\oint_k [D^{-1}(\phi; k)G_0(k) - 1] \\ + V_2^{(a)}(G_0) + V_2^{(b)}(G_0) + V_2^{(c)}(G_0) + V_2^{(d)}(G_0) , \quad (1)$$

where

$$V_2^{(a)}(G) \equiv -\frac{\lambda}{4!}\oint_p \oint_q [G_{aa}(p)G_{bb}(q) + 2G_{ab}(p)G_{ba}(q)] , \quad (2)$$

$$V_2^{(b)}(G) \equiv -\frac{e^2}{2}g^{\mu\nu}\oint_p \oint_q G_{\mu\nu}(q)G_{aa}(p) , \quad (3)$$

$$V_2^{(c)}(G) \equiv \frac{e^2}{4}\oint_p \oint_q \epsilon_{ab}\epsilon_{cd}(2p+q)^\mu(2p+q)^\nu G_{\mu\nu}(q)G_{ad}(p)G_{bc}(p+q) , \quad (4)$$

$$V_2^{(d)}(G) \equiv \frac{e^2}{2}\oint_p \oint_q \epsilon_{ac}\epsilon_{db}(2q+p)^\mu(2q+p)^\nu G_{ab}(p+q)G_{cv}(p)G_{\mu d}(q) . \quad (5)$$

D is the tree-level propagator in momentum space, which can be written as

$$\begin{aligned}
(D^{-1}(\phi; k))_{\mu\nu} &= (e^2\phi^2 - k^2)\left(\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu}\right) + \left(\frac{k^2}{\xi} - e^2\phi^2\right)\frac{k_\mu k_\nu}{k^2} \\
(D^{-1}(\phi; k))_{ab} &= \left(\frac{\lambda\phi^2}{2} - m^2\right)\delta_{a1}\delta_{b1} + \left(\frac{\lambda\phi^2}{6} - m^2\right)\delta_{a2}\delta_{b2} - \delta_{ab}k^2 \\
(D^{-1}(\phi; k))_{a\mu} &= -iek_\mu\epsilon_{ab}\phi_b,
\end{aligned} \tag{6}$$

and G_0 is the solution of

$$\frac{\delta V_T^{res}(\phi, G)}{\delta G} = 0. \tag{7}$$

The analytic solution of Eq.(7) is beyond our present technical capabilities, and, as a consequence, we cannot evaluate V_T^{res} exactly, unless we resort to numerical methods.

I study V_T^{res} analytically using the observation that an approximate solution of the variational problem (1)-(7) can be obtained by evaluating $V_T^{res}(\phi, G)$ with specific parameter-dependent expressions for $G(k)$ and then varying these parameters. This type of procedure is known¹⁰ as the ‘‘Rayleigh-Ritz variational approximation’’. As the parameter-dependent $G(k)$ I take the following expressions

$$\begin{aligned}
G_{\mu\nu}^{-1} &= (M_t^2 - k^2)t_{\mu\nu}(k) + (M_l^2 - k^2)l_{\mu\nu}(k) + \left(\frac{k^2}{\xi} - e^2\phi^2\right)\frac{k_\mu k_\nu}{k^2}, \\
G_{ab}^{-1} &= \delta_{a1}\delta_{b1}(M_\phi^2 - k^2) + \delta_{a2}\delta_{b2}(M_\chi^2 - k^2), \\
G_{a\mu}^{-1} &= -iek_\mu\epsilon_{ab}\phi_b.
\end{aligned} \tag{8}$$

where $t_{\mu\nu}$ and $l_{\mu\nu}$ are defined by

$$t_{\mu\nu}(k) \equiv \delta_{\mu i}\delta_{\nu j}(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}), \quad l_{\mu\nu}(k) \equiv \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} - t_{\mu\nu}. \tag{9}$$

The Eqs.(8) express the propagator in terms of ‘‘Rayleigh-Ritz effective masses’’ M_x . [As required by the way Lorentz invariance is broken at finite temperature¹¹, the two transverse modes of the gauge boson acquire the same effective mass M_t whereas the longitudinal mode has an independent effective mass M_l .]

The approximation of the daisy and superdaisy resummed finite temperature effective potential that I evaluate is the solution of the following variational problem

$$V_T^{res} \simeq V_T^{res}(\phi, G(\{M_0\})), \tag{10}$$

$$\left[\frac{\delta V_T^{res}(\phi, G(\{M\}))}{\delta M^n} \right]_{\{M\}=\{M_0\}} = 0, \tag{11}$$

where $\{M\} \equiv \{M^1, M^2, M^3, M^4\} \equiv \{M_\phi, M_\chi, M_t, M_l\}$.

The effective potential $V_T^{res}(\phi, G(\{M\}))$ in Eqs.(10)-(11) includes divergent integrals; therefore a regularization and renormalization procedure is necessary. In the

similar renormalization of the $\lambda\Phi^4$ scalar theory⁷ it has been shown that the only effect of renormalization on the high-temperature part of the effective potential is the substitution of the bare parameters with renormalized ones. In the following I shall assume that the same applies in the case of the Abelian Higgs Model, and therefore, rather than performing renormalization explicitly, I shall simply omit the (zero-temperature) ultraviolet-divergent contributions and substitute the bare parameters with renormalized ones in my high-temperature effective potential.

Using the well-known results¹²

$$\begin{aligned}\oint_k \ln[k^2 - y^2] &\simeq -\frac{\pi^2 T^4}{45} + \frac{y^2 T^2}{12} - \frac{y^3 T}{6\pi} + \frac{c_\Omega y^4}{16\pi^2}, \\ \oint_k \frac{1}{k^2 - y^2} &\simeq \frac{T^2}{12} - \frac{T y}{4\pi} + \frac{c_\Omega}{8\pi^2} y^2, \\ c_\Omega &\equiv \frac{1}{2} \ln\left(\frac{T^2}{\mu^2}\right) + \frac{1}{2} + \ln(4\pi) - \gamma_{Euler},\end{aligned}\quad (12)$$

(where μ is a renormalization scale), and the high-temperature approximation of $V_2^{(c)}$ obtained in Ref.8, one can easily show that, for $M_x^2/T^2 \ll 1$, $V_T^{res}(\phi, G(\{M\}))$ can be approximated by

$$\begin{aligned}V_T^{res}(\phi, G(\{M\})) &\simeq -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{T^2}{24}\left(\frac{3}{2}\lambda\phi^2 - 2m^2 + 3e^2\phi^2\right) \\ &+ \frac{T}{24\pi}(M_\phi^3 + M_\chi^3 + 2M_t^3 + M_l^3) \\ &- \frac{c_\Omega}{32\pi^2}(M_\phi^4 + M_\chi^4 + 2M_t^4 + M_l^4) \\ &+ \frac{e^2 T^2}{32\pi^2}(M_t^2 - 2M_\phi^2 - 2M_\chi^2) \ln\left(\frac{M_t + M_\phi + M_\chi}{3T}\right) \\ &- \frac{e^2 T \phi^2}{8\pi}(2M_t + M_l) - \frac{T}{8\pi}\left[M_\phi\left(\frac{\lambda_\phi^2}{2} - m^2\right) + M_\chi\left(\frac{\lambda_\phi^2}{6} - m^2\right)\right] \\ &- \frac{e^2 T^3}{24\pi}M_l - \left(\frac{\lambda}{144\pi} + \frac{e^2}{32\pi^2}\right)T^3(M_\phi + M_\chi) \\ &+ e^2 T^2\left[\frac{a_\Phi}{16\pi}(M_\phi^2 + M_\chi^2) - \frac{c_{\Theta t}}{128\pi^2}M_t^2 + \left(\frac{c_\Omega}{48\pi^2} - \frac{c_{\Theta l}}{128\pi^2}\right)M_l^2\right] \\ &+ \frac{e^2 T^2}{32\pi^2}(M_l + M_t)(M_\phi + M_\chi) + \left(\frac{e^2}{32} + \frac{\lambda}{192}\right)\frac{T^2}{\pi^2}M_\phi M_\chi, \quad (13)\end{aligned}$$

where $a_\Phi \equiv [(c_\Omega + c_{\Theta t} + c_{\Theta l})/(4\pi)] + \lambda[(4c_\Omega + 9)/(72\pi e^2)]$. $c_{\Theta t}$ and $c_{\Theta l}$, which are coefficients analogous to c_Ω , are given by integrals that can be evaluated numerically^{8,13}.

The approximation of $V_T^{res}(\phi, G(\{M\}))$ obtained in Eq.(13) allows to express the gap equations (11) in the following high-temperature form

$$\begin{aligned}M_{\phi(\chi)}^2 &\simeq m_{\phi(\chi)}^2 + \left(\frac{\lambda}{18} + \frac{e^2}{4}\right)T^2 - \left[a - \frac{1}{4\pi} - \frac{1}{\pi} \ln\left(\frac{M_\phi + M_\chi}{3T}\right)\right]e^2 T M_{\phi(\chi)} \\ &- \frac{\lambda}{24\pi}T M_{\chi(\phi)} - \frac{e^2}{4\pi}T M_l + \frac{c_\Omega}{\pi} \frac{M_{\phi(\chi)}^3}{T},\end{aligned}\quad (14)$$

$$M_t^2 \simeq e^2 \phi^2 + \left[\frac{c_{\Theta t}}{16\pi} - \frac{1}{8\pi} - \frac{1}{4\pi} \ln\left(\frac{M_\phi + M_\chi}{3T}\right) \right] e^2 T M_t + \frac{c_\Omega}{\pi} \frac{M_t^3}{T}, \quad (15)$$

$$M_l^2 \simeq e^2 \phi^2 + \frac{e^2}{3} T^2 - e^2 \left(\frac{c_\Omega}{3\pi} - \frac{c_{\Theta l}}{8\pi} \right) T M_l - \frac{e^2 T}{4\pi} (M_\phi + M_\chi) + \frac{c_\Omega}{\pi} \frac{M_l^3}{T}. \quad (16)$$

Finally, reexpressing some terms in Eq.(13) using the gap equations (14)-(16), I find that the Rayleigh-Ritz and high-temperature approximation of the daisy and superdaisy resummed finite temperature effective potential for the Abelian Higgs Model is given by

$$\begin{aligned} V_T^{res}(\phi, \{M_0\}) \simeq & -\frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{T^2}{24} \left(\frac{2}{3} \lambda \phi^2 - 2m^2 + 3e^2 \phi^2 \right) \\ & - \frac{T}{12\pi} (M_{\phi,0}^3 + M_{\chi,0}^3 + 2M_{t,0}^3 + M_{l,0}^3) \\ & + \frac{e^2 T^2 (2M_{\phi,0}^2 + 2M_{\chi,0}^2 - M_{t,0}^2)}{32\pi^2} \ln\left(\frac{M_{\phi,0} + M_{\chi,0}}{3T}\right) \\ & + \frac{3c_\Omega}{32\pi^2} (M_{\phi,0}^4 + M_{\chi,0}^4 + 2M_{t,0}^4 + M_{l,0}^4) \\ & - \frac{e^2 T^2}{32\pi^2} M_{l,0} (M_{\phi,0} + M_{\chi,0}) + \left(\frac{e^2}{32} - \frac{\lambda}{192} \right) \frac{T^2}{\pi^2} M_{\phi,0} M_{\chi,0} \\ & - \frac{e^2 T^2}{\pi^2} \left(\frac{c_\Omega}{48} + \frac{c_{\Theta t}}{128} \right) M_{l,0}^2 + \tilde{a}_\Phi \frac{e^2 T^2}{\pi^2} (M_{\phi,0}^2 + M_{\chi,0}^2) \\ & + \frac{c_{\Theta t}}{128\pi^2} e^2 T^2 M_{t,0}^2, \end{aligned} \quad (17)$$

where $M_{\phi,0}$, $M_{\chi,0}$, $M_{t,0}$, and $M_{l,0}$ are the solutions of the gap equations (14)-(16), and $\tilde{a}_\Phi \equiv 1/32 - (c_{\Theta t} + c_{\Theta l})/64 + (\lambda/e^2)(c_\Omega/288 - 1/128)$.

Concerning the nature of the phase transition of the Abelian Higgs Model it is useful to notice that for $eT \ll \phi \ll T$ the Eqs.(14)-(17) imply that: (I) besides the expected contributions involving even powers of ϕ , there is a negative contribution of order $e^3 T \phi^3$ to the effective potential, which comes from the $T M_{t,0}^3$ term, and (II) there is no contribution of order $e^3 T^3 \phi$. These observations indicate^{9,14} that there is a critical temperature T_c at which $V_T^{res}(\phi)$ has two degenerate minima. From Eqs.(14)-(17) it is also easy to realize that when $e^2/\lambda \gg 1$ the symmetry breaking minimum ϕ_b is located in the region of the ϕ -axis that is reliably described by the daisy and superdaisy resummed effective potential*, i.e. $\phi_b > eT_c$ (see Fig.1), and therefore, at least in these hypotheses, my result indicates that the Abelian Higgs Model has a first order phase transition.

Another interesting aspect of Eq.(17) is that the terms linear in the effective masses have cancelled out. In the literature there has been an extensive debate on the possibility that the resummation of the daisy and superdaisy diagrams might induce contributions to the finite temperature effective potential which are linear

*As discussed in Refs.8,9,14, the daisy and superdaisy resummed effective potential is expected to give a reliable approximation of the full effective potential for all $\phi > eT$.

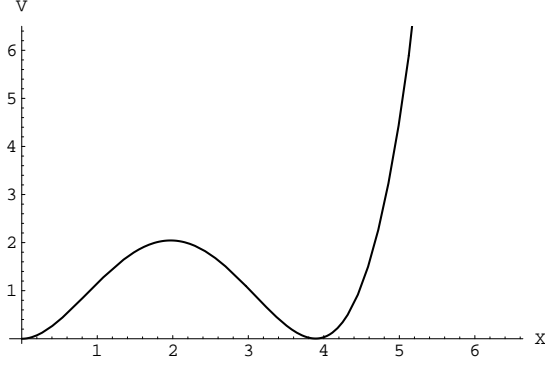


Figure 1: The Rayleigh-Ritz and high-temperature approximation of the daisy and superdaisy resummed effective potential at the phase transition. In figure $V(X) \equiv 10^5 \text{Re}[V_T^{\text{res}}(X) - V_T^{\text{res}}(0)]/T^4$, $X \equiv \phi/eT$, $T \simeq T_c \simeq 8.625m$, $e = .24$, $\lambda = 0.01$.

in the effective masses[†]. Using the general form of the Rayleigh-Ritz approximation with momentum independent effective masses it is easy to see that a cancellation of linear terms always occurs. Let me consider for simplicity a purely bosonic theory with N fields, whose tree-level masses I label m_i ($i=1\dots N$). For such a theory the Rayleigh-Ritz approximation of the effective potential (with appropriate parametrization of G in terms of effective masses M_i) takes the general form

$$\begin{aligned}
V_{R-R} &= V_{\text{classic}}(\phi) + \frac{1}{2} \oint_k \ln G^{-1}(\{M\}; k) \\
&\quad + \frac{1}{2} \oint_k [D^{-1}(\phi; k) G(\{M\}; k) - 1] + V_2(\phi, \{M\}) \\
&= V_{\text{classic}}(\phi) + \sum_i \left(\frac{T^2 M_i^2}{24} - \frac{T M_i^3}{12\pi} + \dots \right) \\
&\quad + \sum_i (m_i^2 - M_i^2) \left(\frac{T^2}{24} - \frac{T M_i}{8\pi} + \dots \right) + V_2(\phi, \{M\}) , \quad (18)
\end{aligned}$$

and the gap equations that follow from varying V_{R-R} have the form

$$M_i^2 = m_i^2 - \frac{8\pi}{T} \frac{\partial V_2(\phi, \{M\})}{\partial M_i} + \dots . \quad (19)$$

Using the gap equations, V_{R-R} can be written as

$$\begin{aligned}
V_{R-R} &= V_{\text{classic}}(\phi) + \sum_i \left(\frac{T^2 m_i^2}{24} - \frac{T M_i^3}{12\pi} + \dots \right) \\
&\quad - M_i \frac{\partial V_2(\phi, \{M\})}{\partial M_i} + V_2(\phi, \{M\}) + \dots . \quad (20)
\end{aligned}$$

[†]More precisely, it has been conjectured that $V_T^{\text{res}} - V_{\text{classic}} - V_{\text{one-loop}}^*$, where $V_{\text{one-loop}}^*$ is the leading one-loop contribution (which, for example, in the case of the Abelian Higgs Model is given by $T^2(2\lambda\phi^2/3 - 2m^2 + 3e^2\phi^2)/24$), might include terms linear in the effective masses.

Contributions linear in M_i can come from the terms $-M_i \partial V_2(\phi, \{M\}) / \partial M_i$ and $V_2(\phi, \{M\})$, but, evidently, each linear contribution coming from $V_2(\phi, \{M\})$ is exactly cancelled by a corresponding contribution coming from $-M_i \partial V_2(\phi, \{M\}) / \partial M_i$, leading to a combined contribution to V_{R-R} that does not include any term linear in M_i . Because the derivation is independent of the specific form of $V_2(\phi, \{M\})$, this result is valid to all orders (i.e. it applies to the full effective potential and any consistent approximation of it), and in particular it applies to the daisy and superdaisy resummed effective potential.

The techniques discussed in this analysis of the Abelian Higgs Model clearly apply to any gauge theory. Using the composite operator method, one can do better than the daisy and superdaisy resummation by going beyond the lowest non-trivial order in the loop expansion of $V(\phi, G)$. Also my Rayleigh-Ritz approximation can be improved by using more elaborated versions of the parameter dependent expression for G ; for example, one can make the substitutions $M_x^2 \rightarrow M_x^2 + Y_x \mathbf{k}^2$ in Eq.(8) and vary not only the M_x 's but also the additional parameters Y_x . Numerical methods can be used both in the study of these more elaborated versions of the Rayleigh-Ritz approximation, and in the exact evaluation of the daisy and superdaisy resummed effective potential as given in Eqs.(1)-(7).

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References

1. R. Takahashi, *Z. Phys.* **C26** (1985) 601.
2. W. Buchmueller and T. Helbig, DESY 92-117 (1992).
3. P. Arnold, *Phys. Rev.* **D46** (1992) 2628.
4. V. Jain, MPI-Ph/92-72 (1992).
5. P. Arnold and O. Espinosa, *Phys. Rev.* **D47** (1993) 3546.
6. A. Hebecker, DESY-93-086 (1993).
7. G. Amelino-Camelia and S.-Y. Pi, *Phys. Rev.* **D47** (1993) 2356.
8. G. Amelino-Camelia, to appear in *Phys. Rev.* **D**.
9. S.D.H. Hsu, in *Proceedings of the Yale/Texas Workshop on (B+L) Violation* (World Scientific, Singapore, 1992); C.G. Boyd, D.E. Brahm and S.D.H. Hsu, CALT-68-1795/HUTP-92-A027/EFI-92-22 (1992).
10. J.M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev.* **D10** (1974) 2428.
11. D.J. Gross, R.D. Pisarski, and L.G. Yaffe, *Rev. Mod. Phys.* **53** (1981) 43.
12. L. Dolan and R. Jackiw, *Phys. Rev.* **D9** (1974) 3320.
13. R.R. Parwani, *Phys. Rev.* **D45** (1992) 4695.
14. M. Dine, R.G. Leigh, P. Huet, A. Linde, and D. Linde, *Phys. Rev.* **D46** (1992) 550; *Phys. Lett.* **B283** (1992) 319.